Biostat 216 Midterm

Nov 20, 2022, 10am-11:50am.

In class, closed-book, one page (double-sided, letter size) cheat sheet allowed.

Make sure to write your **name** and **UID** on your answer sheets. Also number the answer sheets.

- Q1. (10pts) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$.
  1. Give the names of the four fundamental subspaces associated with $\mathbf{A}$.
  2. Pick two (fundamental) subspaces that are essentially disjoint. Give a proof of that they are essentially disjoint.
- Q2. (5pts). Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_k\}$ be a basis of a vector space $\mathcal{S}$. Show that any vector $\mathbf{x} \in \mathcal{S}$ can be expressed **uniquely** as a linear combination of vectors in $\mathcal{A}$.
- Q3. (15pts) Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix with rank $r > 0$.
  1. Show the existence of a rank factorization $\mathbf{A} = \mathbf{CR}$. What are the dimensions of $\mathbf{C}$ and $\mathbf{R}$?
  2. Use the existence of a rank factorization to show that the row rank of $\mathbf{A}$ is equal to the column rank of $\mathbf{A}$.
  3. Matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ has entries $a_{ij} = j^2$. Write down a rank factorization $\mathbf{A} = \mathbf{CR}$ (write down $\mathbf{C}$ and $\mathbf{R}$). What is the rank of $\mathbf{A}$? What is the dim($\mathcal{N}(\mathbf{A})$)? What is the row rank of $\mathbf{A}$?
- Q4. (8pts) A few flop count problems. For flop count, you don't need to derive it and just giving the dominant term (e.g., $2mn$) is fine.
  1. *Inner product*. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$, how many flops does it take to compute $\mathbf{a}'\mathbf{b}$?
  2. *Axpy*. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, how many flops does it take to compute $\alpha\mathbf{x} + \mathbf{y}$?
  3. *Matrix-vector multiplication*. For $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{x} \in \mathbb{R}^n$, how many flops to compute $\mathbf{Ax}$?
  4. *Matrix-matrix multiplication*. For $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times p}$, how many flops to compute $\mathbf{AB}$?
  5. *Quadratic form*. For $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{x} \in \mathbb{R}^m$, and $\mathbf{y} \in \mathbb{R}^n$, how many flops to compute $\mathbf{x}'\mathbf{Ay}$?
  6. *Multiplication by rank-1 matrix*. For $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} = \mathbf{xy}' \in \mathbb{R}^{n \times p}$. How many flops to compute $\mathbf{AB}$ **efficiently**?
  7. *Gram-Schmidt algorithm*. For $\mathbf{A} \in \mathbb{R}^{n \times k}$, suppose the Gram-Schmidt algorithm successfully completes after k iterations. How many flops does the algorithm take?
  8. *L2 norm*. How many flops to compute the norm of a vector $\mathbf{x} \in \mathbb{R}^n$?
- Q5. (5pts) Let $\mathbf{a}_1, \ldots, \mathbf{a}_k \in \mathbb{R}^n$ be a set of orthonormal vectors. Show that they are linearly independent.
- Q6. (15pts) Let $\mathbf{C}$ be a stacked matrix

$$\mathbf{C} = \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}.$$

  1. If columns of $\mathbf{B}$ are linearly independent, can we conclude that columns of $\mathbf{C}$ are linearly

independent?

2. What is the relation between the null space of $\mathbf{C}$ and the null spaces of $\mathbf{A}$ and $\mathbf{B}$. Prove your claim.
3. What is the relation between the rank of $\mathbf{C}$ and the rank of $\mathbf{A}$? Prove your claim.

- Q7. (9pts) Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two vector spaces in $\mathbb{R}^n$. Which of the following are vector spaces? For the first three sets, just indicate each of them is a vector space or not (without proof). For the last set $\mathcal{S}_1^\perp$, either show this is a always vector space or find a counter-example to show that it may not be a vector space.
    - [] $\mathcal{S}_1 \cap \mathcal{S}_2$.
    - [] $\mathcal{S}_1 \cup \mathcal{S}_2$.
    - [] $\mathcal{S}_1 + \mathcal{S}_2$.
    - [] $\mathcal{S}_1^\perp = \{\text{all vectors in } \mathbb{R}^n \text{ that are orthogonal to every vector in } \mathcal{S}_1\}$.
- Q8. (5pts) Let $f : \mathbb{R}^3 \mapsto \mathbb{R}$ be defined as

$$f(\mathbf{x}) = f(x_1, x_2, x_3) = e^{2x_1 + x_2} - x_1 + x_2^2.$$

1. What is the gradient of f? Evaluate the gradient at point $\mathbf{0}_3$.
2. What is the first-order Taylor approximation, or affine approximation, of f at point $\mathbf{0}_3$?

- Q9. (10pts) Recall that the trace function is the sum of diagonal entries of a square matrix.

1. For $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times m}$, show that
$$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA}).$$
This result is notable for the fact that $\mathbf{AB} \neq \mathbf{BA}$ in general.
2. Show the more general result that the trace is invariant under cyclic permutations. For example
$$\text{tr}(\mathbf{ABC}) = \text{tr}(\mathbf{BCA}) = \text{tr}(\mathbf{CAB}).$$
Here we assume that the matrices have compatible dimensions such that all matrix products make sense.

- Q10. (10pts).

1. (5pts) Show that the Gram matrix $\mathbf{A}'\mathbf{A}$ has the same null space as $\mathbf{A}$.
2. (2pts) State the rank-nullity theorem. (You don't need to show it).
3. (3pts) Show that $\text{rank}(\mathbf{A}'\mathbf{A}) = \text{rank}(\mathbf{A})$ (fundamental theorem of ranks).